Diffusive alignment of the magnetic field in active regions of plasmas

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Regions of high magnetic field within plasmas tend to keep this field aligned in a dominant direction. This occurs both in observed phenomena and in simulations of kinematic and nonlinear dynamos. Although most of this effect is due to the particular dynamics of each case, magnetic diffusion also plays an important role. It is shown here that the unitary magnetic field vector satisfies a certain estimate that bounds its possible variations. The dependence of the bound on the plasma parameters is analyzed.

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I. INTRODUCTION

Although the details of the process of growth and maintenance of magnetic fields within charged fluids are not yet fully understood, we have a reasonable picture of the mechanism of transport by fluid motions and dissipation by resistivity to account for many observed features of real plasmas. Among the most consistent characteristics one finds is that the magnetic field tends to be parallel (sometimes antiparallel) in regions where its magnitude is large. Sunspots are a typical example: the magnetic field either points outward or inward from the Sun's atmosphere, and it does not disperse around. Sunspots of opposite polarity may be close together, but still the radial character of the field is predominant. Simulations of dynamos are plentiful: most of them take the fluid velocity for granted, disregarding the effect of the Lorentz force upon it (kinematic dynamo), but increasingly models take into account the full magnetohydrodynamic equations or some nonlinear approximation to them. The phenomenon of alignment is also present: the most active regions are often cigar- or sheet-shaped, and in each of them the magnetic field is essentially unidirectional (see, e.g., [1,2]). These facts admit *ad hoc* explanations: if, as universally admitted, sunspots are the regions where a magnetic flux tube erupts through the surface, it is natural for the field to follow the tube axis; as for dynamo simulations, the magnetic field concentrates in regions of constructive folding (see, e.g., [3]) and follows the most unstable direction of the flow [4], which usually is more or less constant in those regions. These arguments, although far from rigorous, seem essentially solid; anyway it appears that most of the magnetic field alignment is due to the dynamics of the phenomenon under consideration. But why does one never see a wide open sheaf of a strong magnetic field starting from a small region? Diffusivity is one reason: sharp changes in the direction of the field are rapidly smoothed, at a quicker rate than the magnitude of the field. We will prove that if we decompose the magnetic field $\mathbf{B}$ in its magnitude B times the unit vector $\mathbf{b}$, $B|\nabla \mathbf{b}|^2$ remains bounded in the mean by a constant depending only on the forcing and the initial conditions. Thus we may guarantee that no dynamic process will produce a large $|\nabla \mathbf{b}|$ in a sizable region where B is also large.

The paper is divided in two main parts. In the first one we deduce from the equations of incompressible magnetohydrodynamics a number of inequalities: although most of them are classical, the wide variety of possible boundary conditions is not often addressed and many of them are important enough to be considered. The second part builds upon the previous estimates to study the behavior of the normalized magnetic field and the parameters affecting it: in addition to the forcing, boundary, and initial conditions, the presence of interfaces of null magnetic field within the plasma plays a positive role, in the sense that they force the field to be more parallel. The size of kinetic and magnetic diffusivity, i.e., of viscosity and resistivity obviously is essential in the main estimates. These bounds have been derived from recent results on fluid vorticity [5,6], but many features are specific to magnetohydrodynamics.

II. BOUNDARY CONDITIONS AND ENERGY INEQUALITIES

We will assume that the plasma satisfies the incompressible magnetohydrodynamics (MHD) equations with a generic forcing:

$$\frac{\partial \mathbf{u}}{\partial t} = \nu \Delta \mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} + \mathbf{B}\cdot\nabla\mathbf{B} - \nabla p - \nabla\left(\frac{B^2}{2}\right) + \mathbf{f}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \Delta \mathbf{B} - \mathbf{u}\cdot\nabla\mathbf{B} + \mathbf{B}\cdot\nabla\mathbf{u} + \mathbf{g}, \tag{2}$$

$$\nabla\cdot\mathbf{u} = \nabla\cdot\mathbf{B} = 0, \tag{3}$$

plus some adequate boundary conditions. $\mathbf{u}$ stands for the fluid velocity, $\mathbf{B}$ is the magnetic field, p is the kinetic pressure, $\mathbf{f}$ and $\mathbf{g}$ are possible forcing terms, ν is the fluid viscosity and η the resistivity of the plasma. With a different scaling ν and η could represent the inverses of the kinetic and magnetic Reynolds numbers. It often happens that $\mathbf{f}\neq\mathbf{0}$, but $\mathbf{g}=\mathbf{0}$, i.e., that while some material force, such as gravitation, acts upon the fluid, there is no added current for the induction equation. In this case the dependence of the estimates upon η improves. In the ideal case $\eta=0$, the magnetic field lines are transported by the flow as material points. The kinematic dynamo problem considers only the induction equation (2), by assuming $\mathbf{u}$ known *a priori*. As for the domain Ω where the fluid lies and the boundary conditions **u** and **B** must satisfy, we will allow many of them, provided a Poincaré inequality holds for the space of functions under consideration. That is, there must be a constant α such that for all **w** satisfying the conditions of our problem,

$$\int_\Omega |\mathbf{w}|^2\, dV \leqslant \alpha \int_\Omega |\boldsymbol{\nabla}\mathbf{w}|^2\, dV. \tag{4}$$

This happens for practically all the interesting cases. If Ω is bounded and rather smooth (Lipschitz), and there exists a continuous seminorm p on the Sobolev space

$$H^1(\Omega)^N = \left\{ \mathbf{w} \Big/ \int_\Omega |\mathbf{w}|^2\, dV + \int_\Omega |\boldsymbol{\nabla}\mathbf{w}|^2\, dV < \infty \right\}, \tag{5}$$

such that p is a norm in the constant functions (only vanishing at the zero function) and it is identically zero on the subspace of $H^1(\Omega)^N$ of functions satisfying our conditions, then the Poincaré inequality always holds [7,8]. For instance, p could be

$$p(\mathbf{w}) = \left| \int_\Omega \mathbf{w}\, dV \right|,$$

$$p(\mathbf{w}) = \int_{\partial\Omega} |\mathbf{w}\cdot\mathbf{n}|\, d\sigma,$$

$$p(\mathbf{w}) = \int_{\Omega_0} |\mathbf{w}|\, dV, \quad \Omega_0 \subset \Omega, Vol(\Omega_0) \neq 0.$$

The first condition is appropriate for spaces of functions of mean zero, the second one for any space such that $\mathbf{w}\cdot\mathbf{n}|_{\partial\Omega} = 0$, and the third one when the functions we are interested in must vanish in a certain subset of Ω. The space $H_0^1(\Omega)^N$ of functions vanishing in the boundary satisfies the Poincaré inequality even if Ω is bounded only in one direction, i.e., when it is contained between two hyperplanes. Periodic problems are studied for functions of zero mean; when the flow does not cross the boundary one has $\mathbf{u}\cdot\mathbf{n} = 0$, and if there is no inflow of plasma, $\mathbf{u}\cdot\mathbf{n} \geqslant 0$. Except for the periodic case we will always assume $\mathbf{B}\cdot\mathbf{n} = 0$ at the boundary, which happens when the magnetic field outside Ω is null or parallel to the boundary. Transmission problems are also allowed: if there is a wall within Ω such that $\mathbf{u}\cdot\mathbf{n}$, $\mathbf{B}\cdot\mathbf{n}$, u^2, B^2 and p are continuous through it, the inequalities will remain valid.

We will make some notational conventions to avoid a cumbersome number of integrals. We will denote by $|\,|$ the $L^2(\Omega)$ (or $L^2(\Omega)^N$) norm of a function

$$|\mathbf{w}| = \left(\int_\Omega |\mathbf{w}(t,\mathbf{x})|^2\, dV(\mathbf{x}) \right)^{1/2}.$$

For a fixed time interval $[0,T]$, $\|\,\|$ will denote the $L^2(\Omega \times [0,T])$ norm:

$$\|\mathbf{w}\| = \left(\int_0^T dt \int_\Omega |\mathbf{w}(t,\mathbf{x})|^2\, dV(\mathbf{x}) \right)^{1/2}.$$

Under any of these conditions, multiplying Eq. (1) by **u**, Eq. (2) by **B**, integrating in Ω and adding we get

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{u}|^2 + \frac{1}{2}\frac{\partial}{\partial t}|\mathbf{B}|^2 + \nu|\boldsymbol{\nabla}\mathbf{u}|^2 + \eta|\boldsymbol{\nabla}\mathbf{B}|^2 \\ &\quad = -\int_{\partial\Omega}\left(p + B^2 + \frac{1}{2}u^2\right)\mathbf{u}\cdot\mathbf{n}\, d\sigma + \frac{\nu}{2}\int_{\partial\Omega}\frac{\partial u^2}{\partial n}\, d\sigma \\ &\qquad + \frac{\eta}{2}\int_{\partial\Omega}\frac{\partial B^2}{\partial n}\, d\sigma + (\mathbf{f},\mathbf{u})_{L^2} + (\mathbf{g},\mathbf{B})_{L^2}, \end{aligned} \tag{6}$$

and integrating in time in an interval $[0,T]$ where we assume all the functions to exist,

$$\begin{aligned} &\frac{1}{2}|\mathbf{u}(T)|^2 + \frac{1}{2}|\mathbf{B}(T)|^2 + \nu\|\boldsymbol{\nabla}\mathbf{u}\|^2 + \eta\|\boldsymbol{\nabla}\mathbf{B}\|^2 \\ &\quad = \frac{1}{2}|\mathbf{u}(0)|^2 + \frac{1}{2}|\mathbf{B}(0)|^2 \\ &\qquad - \int_0^T dt \int_{\partial\Omega}\left(p + B^2 + \frac{1}{2}u^2\right)\mathbf{u}\cdot\mathbf{n}\, d\sigma \\ &\qquad + \frac{\nu}{2}\int_0^T dt \int_{\partial\Omega}\frac{\partial u^2}{\partial n}\, d\sigma \\ &\qquad + \frac{\eta}{2}\int_0^T dt \int_{\partial\Omega}\frac{\partial B^2}{\partial n}\, d\sigma + \int_0^T (\mathbf{f},\mathbf{u})_{L^2}\, dt \\ &\qquad + \int_0^T (\mathbf{g},\mathbf{B})_{L^2}\, dt. \end{aligned} \tag{7}$$

The boundary integrals are negative when $\mathbf{u}\cdot\mathbf{n} \geqslant 0$ (there is no inflow toward the domain), and the size of u^2 and B^2 decreases in mean toward the boundary of Ω. They vanish in periodic problems, for homogeneous Dirichlet and Neumann problems, for the perfect conductor boundary $\mathbf{B}\cdot\mathbf{n} = 0$, $(\boldsymbol{\nabla} \times \mathbf{B}) \times \mathbf{n} = \mathbf{0}$, and they are negative for mixed problems without intake of plasma: If $\mathbf{u}\cdot\mathbf{n} \geqslant 0$ and A is a positive matrix such that $(A\mathbf{w} + \partial\mathbf{w}/\partial n)|_{\partial\Omega} = \mathbf{0}$, then

$$\int_{\partial\Omega}\frac{\partial w^2}{\partial n}\, d\sigma = -2\int_\Omega \mathbf{w}\cdot A\mathbf{w}\, dV \leqslant 0.$$

Not only in these configurations is the boundary integral never positive: in most realistic cases, the size of velocity and magnetic field decreases in mean towards the boundary, so we will assume from now on that the boundary terms are not positive. Using Cauchy-Schwarz's inequality, one gets

$$\begin{aligned} &\tfrac{1}{2}|\mathbf{u}(T)|^2 + \tfrac{1}{2}|\mathbf{B}(T)|^2 + \nu\|\boldsymbol{\nabla}\mathbf{u}\|^2 + \eta\|\boldsymbol{\nabla}\mathbf{B}\|^2 \\ &\quad \leqslant \tfrac{1}{2}|\mathbf{u}(0)|^2 + \tfrac{1}{2}|\mathbf{B}(0)|^2 + \|\mathbf{f}\|\,\|\mathbf{u}\| + \|\mathbf{g}\|\,\|\mathbf{B}\|. \end{aligned} \tag{8}$$

Since we assume that **u** and **B** satisfy a Poincaré inequality (with possible different constants α and β), we have

$$|\mathbf{u}| \leqslant \alpha|\boldsymbol{\nabla}\mathbf{u}|, \quad |\mathbf{B}| \leqslant \beta|\boldsymbol{\nabla}\mathbf{B}|,$$

$$\|\mathbf{f}\|\,\|\mathbf{u}\| \leqslant \frac{\nu}{2}\|\boldsymbol{\nabla}\mathbf{u}\|^2 + \frac{\alpha^2}{2\nu}\|\mathbf{f}\|^2,$$

$$\|\mathbf{g}\|\,\|\mathbf{B}\| \leqslant \frac{\eta}{2}\|\nabla\mathbf{B}\|^2 + \frac{\beta^2}{2\eta}\|\mathbf{g}\|^2,$$

we find

$$|\mathbf{u}(T)|^2 + |\mathbf{B}(T)|^2 + \nu\|\nabla\mathbf{u}\|^2 + \eta\|\nabla\mathbf{B}\|^2 \leqslant |\mathbf{u}(0)|^2 + |\mathbf{B}(0)|^2 + \frac{\alpha^2}{\nu}\|\mathbf{f}\|^2 + \frac{\beta^2}{\eta}\|\mathbf{g}\|^2. \quad (9)$$

This inequality yields a double estimate: upon the norm of $\nabla\mathbf{u}$, $\nabla\mathbf{B}$ in $L^2([0,T]\times\Omega)$ and upon the norm of $\mathbf{u}(T)$, $\mathbf{B}(T)$ in $L^2(\Omega)$. The last one could be further refined using Gronwall's inequality [8], but we will not need this. If $\mathbf{g}=\mathbf{0}$, we could admit even $\eta=0$ and bound $\mathbf{B}(T)$, but we cannot be certain that $\nabla\mathbf{B}$ remains bounded. The physical meaning is that a magnetic field transported by a possibly chaotic flow and unsmoothed by diffusion may undergo sharp spatial variations, although its energy will not exceed the original one and the one contributed by the flow.

In the kinematic dynamo problem (with $\mathbf{g}=\mathbf{0}$, as it is usually studied) we cannot cancel the term $\int_\Omega(\mathbf{B}\cdot\nabla\mathbf{B})\cdot\mathbf{u}\,dV$ with $\int_\Omega(\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{B}\,dV$, and we are left with

$$\frac{1}{2}\frac{\partial}{\partial t}|\mathbf{B}|^2 + \eta|\nabla\mathbf{B}|^2 = \frac{\eta}{2}\int_{\partial\Omega}\frac{\partial B^2}{\partial n}\,d\sigma + \int_\Omega(\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{B}\,dV. \quad (10)$$

Assuming as before that the boundary integral is not positive and denoting by $|\nabla\mathbf{u}|_\infty$ the supremum of $\nabla\mathbf{u}$ in Ω, we have

$$\frac{1}{2}\frac{\partial|\mathbf{B}|^2}{\partial t} \leqslant |\mathbf{B}|^2\left(|\nabla\mathbf{u}|_\infty - \frac{\eta}{\beta}\right), \quad (11)$$

so that

$$|\mathbf{B}(t)|^2 \leqslant |\mathbf{B}(0)|^2 \exp\left(-\frac{2\eta}{\beta}t + 2\int_0^t|\nabla\mathbf{u}(s)|_\infty\,ds\right). \quad (12)$$

Going back to Eq. (10) and integrating in $[0,T]$,

$$\eta\|\nabla\mathbf{B}\|^2 \leqslant \frac{1}{2}|\mathbf{B}(0)|^2 + \int_0^T|\nabla\mathbf{u}(t)|_\infty|\mathbf{B}(t)|^2\,dt, \quad (13)$$

which may be bounded in several ways by using the previous estimate on $|\mathbf{B}(t)|$. Hence we are able to bound $\|\nabla\mathbf{B}\|$ only by using the additional hypothesis $|\mathbf{u}|_\infty<\infty$.

III. ESTIMATES FOR THE UNIT MAGNETIC FIELD

In every point where the magnetic field does not vanish, let us write $\mathbf{B}=B\mathbf{b}$, where B is the Euclidean norm of $\mathbf{B}$ and $\mathbf{b}$ is a unit vector. Since $\mathbf{B}\cdot\mathbf{B}=B^2$,

$$\frac{1}{2}\frac{\partial B^2}{\partial t} = \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t},$$

$$\tfrac{1}{2}\mathbf{u}\cdot\nabla B^2 = (\mathbf{u}\cdot\nabla\mathbf{B})\cdot\mathbf{B},$$

$$\Delta B^2 = 2(\Delta\mathbf{B})\cdot\mathbf{B} + 2|\nabla\mathbf{B}|^2.$$

Therefore B^2 satisfies in the open set of points where $\mathbf{B}\neq\mathbf{0}$,

$$\frac{1}{2}\left(\frac{\partial B^2}{\partial t} + \mathbf{u}\cdot\nabla B^2 - \eta\Delta B^2\right) + \eta|\nabla\mathbf{B}|^2 = (\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{B} + \mathbf{g}\cdot\mathbf{B}. \quad (14)$$

Since also $\Delta B^2 = 2(\Delta B)B + 2|\nabla B|^2$,

$$B\frac{\partial B}{\partial t} + B\mathbf{u}\cdot\nabla B - \eta B\Delta B - \eta|\nabla B|^2 + \eta|\nabla\mathbf{B}|^2 = (\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{B} + \mathbf{g}\cdot\mathbf{B}. \quad (15)$$

Also, at any point with $\mathbf{B}\neq\mathbf{0}$,

$$|\nabla\mathbf{B}|^2 = \sum_{i,j}(\partial_i B_j)^2 = \sum_{i,j}(\partial_i B)^2(b_j)^2 + B^2\sum_{i,j}(\partial_i b_j)^2 + 2\sum_{i,j}B(\partial_i B)b_j(\partial_i b_j) = |\nabla B|^2 + B^2|\nabla\mathbf{b}|^2,$$

because $\Sigma_j b_j^2 = 1$. Hence, dividing Eq. (15) by B,

$$\frac{\partial B}{\partial t} + \mathbf{u}\cdot\nabla B - \eta\Delta B + \eta B|\nabla\mathbf{b}|^2 = (\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{b} + \mathbf{g}\cdot\mathbf{b}. \quad (16)$$

Although B^2 is at least as smooth as $\mathbf{B}$, B may fail to be differentiable at the null points of $\mathbf{B}$, and the equation above is only valid at the open set $\Omega-\{\mathbf{B}=\mathbf{0}\}$. We will assume $\mathbf{B}$ smooth enough in $\Omega\times[0,T]$, and for the sake of simplicity, that the second differential of $\mathbf{B}$ does not vanish at any null point of $\mathbf{B}$. Then the null sets of $\mathbf{B}$ are either isolated points, lines, or surfaces. For points or curves, $\Delta\mathbf{B}$ is integrable in a neighborhood of them (as one sees by using spherical or cylindrical coordinates: the possible singularity of $\Delta\mathbf{B}$ is integrable). By taking small neighborhoods, we see that these sets may be ignored when integrating Eq. (16) in Ω. The surfaces of null $\mathbf{B}$, however, must be dealt with separately; they are interfaces separating the connected components of $\Omega_0=\Omega-\{\mathbf{B}=\mathbf{0}\}$. The term $\mathbf{u}\cdot\nabla B$ remains bounded throughout Ω even at the sets $B=0$. Moreover, for every component of Ω_0, its boundary is formed in part by surfaces where $B=0$, and therefore the integral of $B\mathbf{u}\cdot\mathbf{n}$ is zero, and (perhaps) by part of $\partial\Omega$. The term $\eta\Delta B$, however, is influenced by interfaces. Let Ω_1 be a component of Ω_0. Then

$$\int_{\Omega_1}\Delta B\,dV = \int_{\partial\Omega_1}\frac{\partial B}{\partial n}\,d\sigma.$$

as before, $\partial\Omega_1-\partial\Omega$ is formed by interfaces where $B=0$: let Γ be one of them. Then Γ is also part of the boundary of another component of Ω_0. For this second region, the outer normal is opposite, while B decreases toward Γ at the same rate at both sides of it (due to the smoothness of B^2). Hence the contribution to the boundary integral of Γ is multiplied by two in the whole integration. If we denote by γ the slope of B in every interface Γ (i.e., $\gamma=\partial B/\partial n$ at both sides), and $\Gamma_j=\Gamma_j(t)$ are all the interfaces,

$$\int_\Omega \Delta B\, dV=2\sum_j \int_{\Gamma_j}\gamma\, d\sigma+\int_{\partial\Omega}\frac{\partial B}{\partial n}\, d\sigma.$$

Notice that since B decreases toward Γ_j, always $\gamma<0$. Integrating Eq. (16) in Ω, we get

$$\frac{\partial}{\partial t}\int_\Omega B\, dV+\eta\int_\Omega B|\nabla \mathbf{b}|^2\, dV$$
$$=-\int_{\partial\Omega}B\mathbf{u}\cdot\mathbf{n}\, d\sigma+2\eta\sum_j\int_{\Gamma_j}\gamma\, d\sigma+\eta\int_{\partial\Omega}\frac{\partial B}{\partial n}\, d\sigma$$
$$+\int_\Omega(\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{b}\, dV+\int_\Omega \mathbf{g}\cdot\mathbf{b}\, dV. \tag{17}$$

Integrating in time in $[0,T]$,

$$\int_\Omega B(T)\, dV+\eta\int_0^T\int_\Omega B|\nabla\mathbf{b}|^2\, dV\, dt$$
$$=-\int_0^T\int_{\partial\Omega}B\mathbf{u}\cdot\mathbf{n}\, d\sigma\, dt+2\eta\int_0^T\sum_j\int_{\Gamma_j}\gamma\, d\sigma\, dt$$
$$+\eta\int_0^T\int_{\partial\Omega}\frac{\partial B}{\partial n}\, d\sigma\, dt+\int_0^T\int_\Omega(\mathbf{B}\cdot\nabla\mathbf{u})\cdot\mathbf{b}\, dV\, dt$$
$$+\int_0^T\int_\Omega\mathbf{g}\cdot\mathbf{b}\, dV\, dt. \tag{18}$$

Notice that the number and shape of Γ_j may depend on t. This first term of the right-hand side is always negative. The second one is negative too, and its size increases with the number of interfaces and their areas, as well as with the rate of decrease of B toward them. If two interfaces are close together and the magnetic field is not small between them, $|\gamma|$ must be large and the contribution of the integral to the right hand side more negative. For intrincate interfaces this term may decrease the bounding constant to a significant degree.

If we assume that B decreases in mean toward the boundary, the third term is also negative. The fourth one may be bounded by $\|\nabla\mathbf{u}\|\,\|\mathbf{B}\|$, and the last term by the integral in $[0,T]\times\Omega$ of $|\mathbf{g}(\mathbf{x})|$. $\|\nabla\mathbf{u}\|$ may be bounded by Eq. (9). This bound scales like ν^{-1} if $\mathbf{g}=\mathbf{0}$, and like $\sup\{\nu^{-1},(\nu\eta)^{-1/2}\}$ otherwise.

As for $\|\mathbf{B}\|$, there are several ways to estimate it. An analogous bound to the previous one using Poincaré's inequality would yield a scale $(\nu\eta)^{-1/2}$ if $\mathbf{g}=\mathbf{0}$, $\sup\{\eta^{-1},(\nu\eta)^{-1/2}\}$ otherwise. We may also forget about the term in $\nabla\mathbf{B}$ and integrate $|\mathbf{B}(T)|$ in time: this would scale like $\sup\{\nu^{-1/2},\eta^{-1/2}\}$, but it would grow in time. These estimates are important because often ν and especially η are very small, so the smaller negative powers of them one uses, the better.

If we are dealing with a kinematic dynamo problem, $\|\nabla\mathbf{u}\|$ must be known to be bounded *a priori*, and $\|\mathbf{B}\|$ is estimated by integrating Eq. (12) in time. In this case η and ν do not occur in the denominator, which is convenient, but the hypothesis of boundedness of the maximum of $|\nabla\mathbf{u}|$ is very strong.

Let M denote the bound obtained in the right-hand term by the above considerations. Notice that it only depends on the initial conditions, the forcing, and the hypothesis of the mean decrease of B toward the boundary. The estimate

$$\int_\Omega B(T)\, dV+\eta\int_0^T\int_\Omega B|\nabla\mathbf{b}|^2\, dV\, dt\leq M \tag{19}$$

is our main result. That $\int_\Omega B(T)\, dV$ is bounded is not a surprise when Ω is itself bounded, because the L^2 norm $|\mathbf{B}(T)|$ is bounded and it dominates the L^1 norm: but for a domain Ω of infinite measure this is an unrelated bound. Essentially it is more demanding on the decrease of $|\mathbf{B}(\mathbf{x})|$ when $|\mathbf{x}|\to\infty$. However, the most interesting term is the second one: it bounds the changes of direction of $\mathbf{B}$ in a more effective way than $\|\mathbf{B}\|$. Its dependence on ν and η is at worst like $\sup\{\eta^{-1}\nu^{-3/2},\eta^{-3/2}\nu^{-1},\eta^{-2}\nu^{-1/2}\}$, and $\eta^{-1}\nu^{-3/2}$ if $\mathbf{g}=\mathbf{0}$.

Generally speaking, this estimate limits the variations of direction of the magnetic field in regions where it is large. As a rather rough example of its order of accuracy, notice that for any $U\subset\Omega$, by Cauchy-Schwarz's inequality,

$$\int_0^T\int_U|\nabla\mathbf{b}|\, dV\, dt\leq M\left(\int_0^T\int_U\frac{1}{B}\, dV\, dt\right)^{1/2}.$$

Thus the mean variation of $\mathbf{b}$ within a subset U where B is in mean larger than L does not exceed the order of $1/\sqrt{L}$.

It is important to notice that when dealing with antiparallel fields close by (say by changing $\mathbf{B}$ to $-\mathbf{B}$ through a null point) one should not interpret $\nabla\mathbf{b}$ as the gradient of a jump (which would yield a distributional δ function, so that the integral of $B|\nabla\mathbf{b}^2|$ would be infinite). Rather this means that $\mathbf{B}$ is crossing an interface and we must integrate separately in each of the regions. Thus it does not matter if $\mathbf{b}$ is parallel or antiparallel, except that the last possibility denotes the presence of interfaces and as explained before decreases the constant M. What is clearly forbidden are rapid spatial variations of $\mathbf{b}$ in regions of large field.

IV. CONCLUSIONS

We have shown that one of the effects of kinetic and magnetic dissipation is the alignment of the magnetic field in active regions of plasmas. The mean in time and space of the field size multiplied by the square of the gradient of the unit field vector is bounded by a constant depending only on the initial conditions and the forcing terms, and improved by the possible presence of surfaces of null magnetic field. The dependence of this constant on the orders of viscosity and resistivity has also been found.

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